Modélisation numérique des guides d'onde tridimensionnels à section symétrique par rotation

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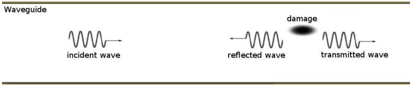
Projet OMCEND : soutien financier et partenaires

Contents

Context

Guided wave applications:

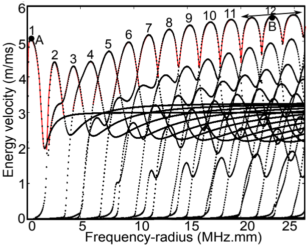
- dynamic analysis of elongated structures
- examples:
  - **N**on **D**estructive **E**valuation (ultrasonics)
  - vibration and noise reduction
  - statistical energy analysis...



NDE: detecting a damage with elastic waves

Generality about waveguides:

- Guided wave propagation: dispersive and multimodal
- Dispersion curves required
- Modeling tools needed



Energy velocity vs. frequency in a cylindrical bar

On the modeling of elastic waveguides

Full 3D approach:

- high frequency (*e.g.* ultrasonics) $\rightarrow$ fine mesh
- guided waves go to infinity $\rightarrow$ large model
- huge computational memory required
- tedious post-processing for wave modes...

Modal approach:

- guided waves = modes
- eigenvalue problem
- plates, cylinders: analytical approaches (Thomson-Haskell, ©Disperse, ...)
- **arbitrary cross section:** finite element discretization
  - of cross-section[1] (often referred to as the "**SAFE**" method)
  - of a 3D slice with Bloch-Floquet periodic conditions[2] ("WFEM")

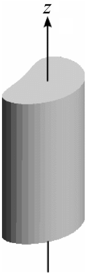
SAFE: **S**emi-**A**nalytical **F**inite **E**lement method
WFEM: Wave Finite Element Method

[1] Lagasse JASA 1973, Aalami JAM 1973, Hayashi et al. Ultrasonics 2003, Bartoli et al. JSV 2006,...
[2] Gry et al. JSV 1997, Mace et al. JASA 2005,...

The SAFE method: accounting for translational invariance

Variational formulation for 3D elastodynamics:

$$\int_\Omega \delta\boldsymbol{\epsilon}^T \mathbf{C}\boldsymbol{\epsilon} \mathrm{d}V + \int_\Omega \rho\delta\mathbf{u}^T\ddot{\mathbf{u}}\mathrm{d}V = 0, \text{ with } \boldsymbol{\epsilon} = (\mathbf{L}_{xy} + \mathbf{L}_z\partial/\partial z)\mathbf{u} \quad (1)$$



3D waveguide of arbitrary cross-section

Perform:

1. **Fourier transform along t and z:**

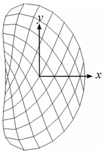
$$\hat{\mathbf{u}}(k,\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{u}(z,t)\mathrm{e}^{-\mathrm{i}(kz-\omega t)}\mathrm{d}z\mathrm{d}t$$

2. **FE discretization of the cross-section (x,y):**

$$\Rightarrow \quad \mathbf{u}(x,y,z,t) = \mathbf{N}^e(x,y)\mathbf{U}^e \mathrm{e}^{\mathrm{i}(kz-\omega t)} \quad (2)$$

Quadratic eigenvalue problem

$$[\mathbf{K}_1 - \omega^2\mathbf{M} + \mathrm{i}k(\mathbf{K}_2 - \mathbf{K}_2^T) + k^2\mathbf{K}_3]\mathbf{U} = \mathbf{0} \quad (3)$$



SAFE mesh

- problem reduced on the cross-section only
- solved for each frequency ω, solution = guided modes $(k_n^\pm, \mathbf{U}_n^\pm)$

The SAFE method: accounting for translational invariance

Notice: $\mathbf{K}_1$, $\mathbf{K}_3$ and $\mathbf{M}$ are symmetric, $(\mathbf{K}_2 - \mathbf{K}_2^{\mathsf{T}})$ is skew-symmetric

Properties of the quadratic eigenvalue problem

- if k_m is an eigenvalue, then $-k_m$ also (take the transpose of (3))
  $\Rightarrow$ pairs of opposite-going modes $\{k_m, \mathbf{U}_m\}$ and $\{-k_m, \mathbf{U}_{-m}\}$
- the orthogonality relation between modes is:

$$\mathrm{i}\frac{\omega}{4}\left(\mathbf{T}^{\mathsf{T}}_{-m'}\mathbf{U}_m - \mathbf{U}^{\mathsf{T}}_{-m'}\mathbf{T}_m\right) = Q_{m,-m'}\delta_{mm'} \tag{4}$$

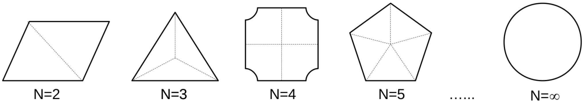
This biorthogonality relation:

- is general (remains applicable for non-propagating modes, fully anisotropic materials and lossy waveguides... no assumptions needed)
- is actually a discrete version of Auld's real biorthogonality relationship[3]
- can be simplified in some particular cases (Auld's complex relation, Fraser's relation[4])

[3] Auld, Acoustic Fields and Waves in Solids, 1990
[4] Fraser, JASA, 1976

Contents

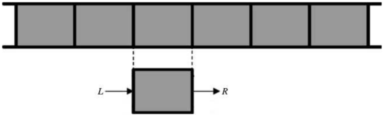
Rotationally symmetric cross-sections: we can do better



Examples of rotational symmetry

Rotational symmetry = circular periodicity

- Reminder: Bloch-Floquet boundary conditions (see *e.g.* Mead JSV 1996)

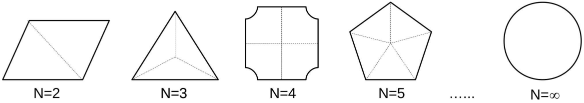


straight periodicity, unit cell

$\mathbf{U}_r = \lambda \mathbf{U}_l, \quad \mathbf{F}_r = -\lambda \mathbf{F}_l$

$\lambda = \mathrm{e}^{\mathrm{i}\mu}$ (iμ: propagation constant)

Rotationally symmetric cross-sections: we can do better



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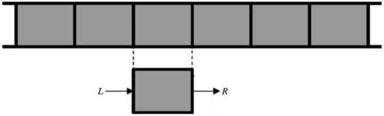

straight periodicity, unit cell

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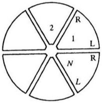
- In case of circular periodicity: $\lambda^N = 1$ (N: order of rotational symmetry)



circular periodicity with N cells

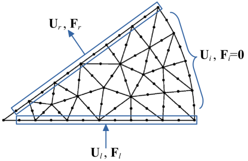
$$\lambda(n) = \mathrm{e}^{\mathrm{i}2n\pi/N}$$

$$n = \begin{cases} -\frac{N}{2}+1, \ldots, 0, \ldots, \frac{N}{2} & \text{for } n \text{ even} \\ -\frac{N-1}{2}, \ldots, 0, \ldots, \frac{N-1}{2} & \text{for } n \text{ odd} \end{cases}$$

Accounting for rotational symmetry in SAFE

1. Partition of dofs in SAFE:

$$\{\mathbf{K}_1 - \omega^2\mathbf{M} + ik(\mathbf{K}_2 - \mathbf{K}_2^T) + k^2\mathbf{K}_3\}\mathbf{U} = \mathbf{F}$$
$$\mathbf{U} = [\mathbf{U}_l^T\ \mathbf{U}_i^T\ \mathbf{U}_r^T]^T \text{ and } \mathbf{F} = [\mathbf{F}_l^T\ \mathbf{F}_i^T\ \mathbf{F}_r^T]^T$$



2. Elasticity variables = $\mathbf{U}$ and $\mathbf{F}$ → **vectorial fields written in a Cartesian frame!**

$$\mathbf{Q}_r\mathbf{U}_r = \lambda\mathbf{Q}_l\mathbf{U}_l \qquad (6a)$$
$$\mathbf{Q}_r\mathbf{F}_r = -\lambda\mathbf{Q}_l\mathbf{F}_l \qquad (6b)$$

$\mathbf{Q}_{l,r}$: transformation matrices from Cartesian to cylindrical frames

3. Build the projection matrix $\mathbf{R}$ from Eq. (6a):

$$\mathbf{U} = \mathbf{R}(n)\tilde{\mathbf{U}}, \quad \mathbf{R}(n) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \lambda(n)\mathbf{Q}_r^{-1}\mathbf{Q}_l & \mathbf{0} \end{bmatrix}, \quad \tilde{\mathbf{U}} = \begin{bmatrix} \mathbf{U}_l \\ \mathbf{U}_i \end{bmatrix}.$$

4. Trick: left multiply SAFE by $\mathbf{R}^*$

$$[\tilde{\mathbf{K}}_1(n) - \omega^2\tilde{\mathbf{M}}(n) + ik(\tilde{\mathbf{K}}_2(n) - \tilde{\mathbf{K}}_2(-n)^T) + k^2\tilde{\mathbf{K}}_3(n)]\tilde{\mathbf{U}} = \mathbf{R}(n)^*\mathbf{F} \qquad \text{with } (\tilde{\cdot}) = \mathbf{R}^*(\cdot)\mathbf{R}$$

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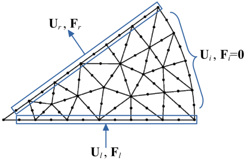

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$$[\tilde{\mathbf{K}}_1(n) - \omega^2\tilde{\mathbf{M}}(n) + ik(\tilde{\mathbf{K}}_2(n) - \tilde{\mathbf{K}}_2(-n)^T) + k^2\tilde{\mathbf{K}}_3(n)]\tilde{\mathbf{U}} = \mathbf{R}(n)^*\mathbf{F} = \mathbf{0} \quad \text{with } \tilde{(\cdot)} = \mathbf{R}^*(\cdot)\mathbf{R}$$

And notice that $\mathbf{R}(n)^*\mathbf{F} = \begin{bmatrix} \mathbf{F}_l + \lambda(n)^*\mathbf{Q}_l^{-1}\mathbf{Q}_r\mathbf{F}_r \\ \mathbf{0} \end{bmatrix} = \mathbf{0}$ from Eq. (6b)

Accounting for rotational symmetry in SAFE

Notice: $\tilde{\mathbf{K}}_1(n)$, $\tilde{\mathbf{K}}_3(n)$ and $\tilde{\mathbf{M}}(n)$ are still symmetric, but...
$(\tilde{\mathbf{K}}_2(n) - \tilde{\mathbf{K}}_2(-n)^{\mathrm{T}})$ is no longer skew-symmetric

Properties of the new quadratic eigenvalue problem

- If k_m is an eigenvalue for the order n, then $-k_m$ also.. but for the order $-n$!
  $\Rightarrow$ pairs of opposite-going modes $\{k_m^{(n)}, \tilde{\mathbf{U}}_m^{(n)}\}$ and $\{-k_m^{(n)}, \tilde{\mathbf{U}}_{-m}^{(-n)}\}$
- The biorthogonality relation between modes is now:

$$\mathrm{i}\frac{\omega}{4}\left(\tilde{\mathbf{T}}_{-m'}^{(-n)\mathrm{T}}\tilde{\mathbf{U}}_m^{(n)} - \tilde{\mathbf{U}}_{-m'}^{(-n)\mathrm{T}}\tilde{\mathbf{T}}_m^{(n)}\right) = Q_{m,-m'}^{(n,-n)}\delta_{mm'} \quad (7)$$

which requires the solution of 2 eigenvalue problems: $+n$ and $-n$

Accounting for rotational symmetry in SAFE

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  which requires the solution of 2 eigenvalue problems: $+n$ and $-n$

... [tedious calculation]... And the forced response in a given cell s is:

$${}^s\tilde{\mathbf{U}} = \sum_n \sum_{m>0} \frac{\mathrm{i}\omega}{4Q_{m,-m}^{(n,-n)}} \tilde{\mathbf{U}}_m^{(n)}\tilde{\mathbf{U}}_{-m}^{(-n)\mathrm{T}}\tilde{\mathbf{F}}_{\mathrm{ext}}^{(n)}(k_m^{(n)})\mathrm{e}^{\mathrm{i}k_m^{(n)}z}\mathrm{e}^{\mathrm{i}\frac{2\pi ns}{N}} \quad \text{with } \mathbf{F}_{\mathrm{ext}}^{(n)} = \frac{1}{N}\sum_{s=0}^{N-1} {}^s\mathbf{F}_{\mathrm{ext}}\mathrm{e}^{-\mathrm{i}\frac{2\pi ns}{N}}$$

Note: degeneracy to the axisym. case (Fourier series)
$\mathrm{e}^{\pm\mathrm{i}\frac{2\pi ns}{N}} \xrightarrow{N\to\infty} \mathrm{e}^{\pm\mathrm{i}n\theta}$, $\frac{1}{N}\sum_{s=0}^{N-1} \xrightarrow{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathrm{d}\theta$

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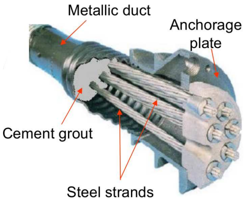
Description of the problem

Main motivation of this work = NDE of cables (damage detection, tension estimation,...)

A common configuration for bridge cables: the seven-wire strand



Anchorage zone of a bridge cable (single access⇒reflectometry)

6+1 strand

Modeling difficulties:

- helical geometry → translational invariance is not straight...
- multiwire waveguides → rotational symmetry frequently occurs

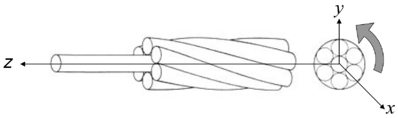
+ prestress and mechanical contact[5] (out of the scope of this talk)

[5] Frikha et al. IJSS 2013, Treyssède JSV 2016

Note on translational invariance: twisting coordinates

Basic assumption for guided waves: $\propto e^{ikz} \rightarrow$ variables must be separable

- translational invariance required
- specific coordinate system needed[6]



Twisting system

- curvilinear coordinate system with zero curvature κ but non zero torsion τ
  $\kappa = 0$, $\tau = 2\pi/L$ (L: period of helical wires)
- the axis z remains straight but the (x, y) plane rotates with z

The differential operators of the SAFE method must be written in this system, *e.g.*:

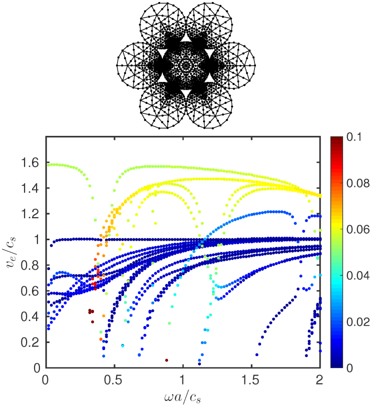
$$\nabla_0 \mathbf{u} = (u_{i,j} - \Gamma^k_{ij} u_k)\mathbf{g}^i \otimes \mathbf{g}^j = \ldots = \begin{bmatrix} u_{x,x} & u_{x,y} & \Lambda u_x - \tau u_y + u_{x,z} \\ u_{y,x} & u_{y,y} & \tau u_x + \Lambda u_y + u_{y,z} \\ u_{z,x} & u_{z,y} & \Lambda u_z + u_{z,z} \end{bmatrix}_{(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)},$$

with $\Lambda = \tau(y(\cdot)_{,x} - x(\cdot)_{,y})$

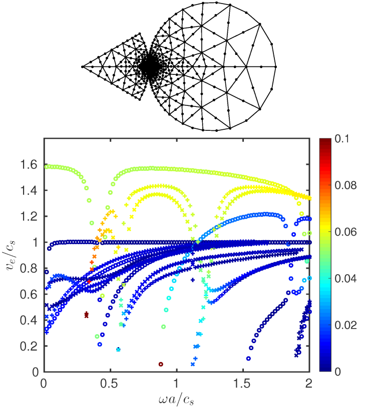
[6] Treysséde and Laguerre JSV 2010

Results: dispersion curves for a seven-wire strand

Steel, a=2.7mm, $\phi = 7.9^\circ$ ($\tau a = 0.0705$), tensile prestress ($e = 0.6\%$), excitation of periph. wire



complete model, 12369 dofs



rotationally symmetric cell (N=6, 2094 dofs)

Energy velocity vs. frequency with modal amplitudes computed with the complete model (grey points), with the reduced model for n=0 ($\circ$), n=1 ($+$) and $n = -1$ ($\times$)

Same results, computational time reduced by a factor $\sim$13 (for a given n)

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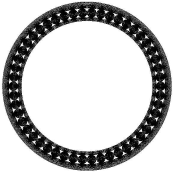
Conclusion: application to more complex cables

architecture of a umbilical power cable

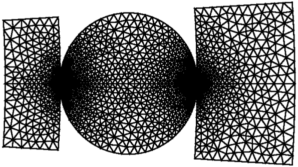


Complete mesh: ~1,000,000 dofs
computation not achievable...

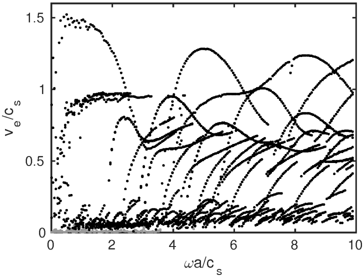
Our goal = NDE of armor

Conclusion: application to more complex cables

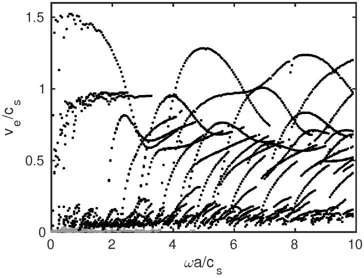
PE-steel-PE, a=2.25mm, ϕ=14°, prestress (0.1% + 5 bars), damping (κ_l=0.02, κ_s=0.16Np/λ)



Rotationally symmetric cell (N=50, 22587 dofs)



Energy velocity vs. frequency for n=0



Energy velocity vs. frequency for $n = N/2$